An essay on an optimal derivative and related dynamic trading strategies

Serguey Khovansky*

University of Virginia

November 16, 2005

Abstract

A payoff of an optimal derivative maximizes expected utility subject to a predetermined price. Replication of the derivative's current value defines an optimal dynamic trading strategy. In this paper I extend the variety of optimal strategies by introducing a new class of their terminal payoffs - a path-dependent optimal derivative. To relate these strategies to those obtained by replicating an optimal path-independent derivative I establish a condition under which such an extension does not increase investor's expected utility. Optimal trading often requires leverage, which might be prohibitively large for a trader with borrowing constraints. I address this problem by constructing an optimal derivative and trading strategy for such an investor. Finally, an approximation of a payoff of an optimal derivative in an incomplete market is suggested and studied in detail in the setting of the Heston stochastic volatility model. This allows for crafting an optimal trading strategy in this framework.

---

*Work in progress. I would like to thank Wake Epps, Massimo Guidolin and participants of FMA conference 2005 for valuable comments and suggestions. Mailing address: Department of Economics, University of Virginia; P.O. Box 400182; Charlottesville, VA, 22904-4182; E-mail: skk9e@virginia.edu

1 Introduction

This paper suggests a new methodology for deriving an optimal trading strategy. The study focuses on two investment vehicles such as an equity stock and a money fund and investigates an optimal trading in this environment. The main contribution comes by exploring the observation that a payoff of every finite-horizon, self-financing trading strategy is a function of the underlying assets. In this sense, every such strategy specifies a financial derivative, and vice verse. This is so because every derivatives's current value can be replicated or approximated[1] by a linear portfolio of stock and money fund shares. Thus, I suggest that in order to identify the best trading strategy an investor first needs to specify a terminal payoff (as a function of the available trading assets) that fits his preferences best and then replicate or approximate the current value of this payoff.

Intuitively, an optimal payoff is characterized by a slight chance for a small adverse outcome and a sizable probability for a large gain. In other words, the larger the gain and the smaller the loss subject to the same initial value the more desirable the payoff is. At the same time, of course, in the setting of competitive markets a terminal payoff permitting an arbitrage opportunity is not viable. In other words, a situation with a possibility for a gain with no possibility for a loss is not allowable.

A payoff found to be optimal by an expected utility maximizer subject to a predetermined initial cost gives rise to an optimal derivative. Obtaining this payoff by way of traded assets generates an optimal trading strategy. Carr and Madan (2001) introduced an optimal *path-independent* derivative and found its payoff for traders with different utilities. They explained how it can be replicated by a static composition of call and put vanilla options along with the money fund and the underlying shares. In this paper I use the optimal payoff to derive not only static but also dynamic optimal trading strategy. Additionally, I extend the idea of an optimal payoff by designing a version of an optimal *path-dependent* derivative. In general, an expected utility maximizer might prefer the latter to the former. If this is the case, then the dynamic trading strategy generated by replicating the current value of a path-independent optimal derivative is not the best investment, unlike the strategy based on the optimal path-dependent structure. However, a path-dependent structure is

[1] Incomplete market setting permits only approximation of a non-linear derivative's value.

not always beneficial. In this paper I find conditions under which a trader does not gain from complicating the payoff structure from path-independent to path-dependent.

Replication of an optimal derivative often requires leverage. However, in practice traders always have limited borrowing capacity. Hence, I modify the optimal derivative's payoff taking into account the fact that an investor faces a borrowing constraint at a certain time.

Finally, I discuss an approximation to an optimal payoff in an incomplete market, taking as an example the Heston stochastic volatility model, in which volatility is not tradable.

This work relates to the research exploring martingale approach to derivative pricing, which was pioneered by Harrison and Kreps (1979) and Harrison and Pliska (1981). It was successfully implemented for problems of optimal intertemporal consumption and portfolio choice in continuous time. In the setting of complete markets it was addressed by Karatzas et al. (1987) and Cox and Huang (1989). For an incomplete market environment it was first solved by He and Pearson (1991). It is important to note that not only investment but also consumption, i.e. fund withdrawal, in these problems were assumed to be continuous. However, in reality, outflow of funds from investment companies occurs at discrete, often prespecified times. To model this situation one requires a framework of continuous stock reinvestment and discrete-time consumption. Generally, the approach of optimal dynamic trading strategies discussed in this paper is able to address this issue.

The fact that the optimal terminal payoff is given by an inverse marginal utility evaluated at the Radon-Nikodym derivative was noted by Leland (1980), Brennan and Solanki (1981) and Pliska (1986). Recent advances can be credited to Zhao (2003) who presented a portfolio with control on the worst case outcome. Bertsimas at el (2001) suggested an approximation of a derivative payoff in incomplete markets.

Merton (1971) demonstrated another way of finding an optimal portfolio which explored techniques of stochastic dynamic programming. His methodology deals with a nonlinear partial differential equation governing utility of the optimized wealth. This approach, however, features several disadvantages. Only special cases admit analytical solutions. Problems with a general utility function or a stochastic process which differs from the geometric Brownian motion are hard to address. Market incompleteness makes the task even more difficult. Questions concerning the form of the terminal payoff were not emphasized at all. How does an investor know which utility describes best his or her preferences if the function of the terminal payoff is not examined? The technique suggested in this paper, on the contrary, readily allows for these complications.

The rest of the paper is organized as follows. The section II acquaints the reader with an idea of a path-independent optimal payoff and explains derivation of the optimal trading strategies. The following section introduces a path-dependent optimal structure and sets conditions for an optimal path-independent derivative to be indeed optimal. Section IV determines an optimal payoff function for an investor who is credit constrained at certain intermediate time. This is done in a complete market setting. Section IV finds the best complete market approximation of an optimal derivative in an incomplete market of Heston stochastic volatility model. Section V provides several concluding remarks and directions for future research.

2 Optimal Path-Independent Derivative

Let $U(\cdot)$ be a utility of a trader with initial wealth W_0 invested in a "stock" worth S_0 at time 0, and a T- expiring bond with the price $B(0,T)$. By assumption, the trader operates in a complete market setting and pursues a self-financing dynamic trading strategy. At the terminal time T the realized payoff of the trading strategy $G(S_T)$ is consumed. The optimal strategy is such that its terminal payoff $G(S_T)$ maximizes expected utility $E_0U(G(S_T))$. Additionally, the fundamental theorem of asset pricing[2] relates the terminal payoff $G(S_T)$ to the initial investment W_0 by establishing, that $B(0,T)\widehat{E_0}G(S_T) = W_0$, where $\widehat{E}$ denotes expectation under the equivalent martingale measure. Such a payoff $G(S_T)$ is referred to as an "optimal path-independent derivative". To summarize, the

[2]The theorem says that under the no arbitrage setting there is an equivalent martingale measure under which the current ratio of two assets equals the conditional expectation of this ratio at any future date. This implies that $\widehat{E}_0\frac{G(S_T)}{M_T} = \frac{G(S_0)}{M_0}$ where M_T is value of a money fund share, given by $M_T = M_0 \exp(rT)$ and r is a risk free interest rate. This equation is equivalent to $\frac{M_0}{M_T}\widehat{E}_0G(S_T) = G(S_0)$. Finally, in this setting the price of the bond is $B(0,T) \equiv \frac{M_0}{M_T}$.

optimal derivative is the solution to the problem

$$\begin{cases} \max\limits_{G} E_0 U(G(S_T)) \\ \text{s.t. } B(0,T)\widehat{E_0}G(S_T) = W_0. \end{cases} \tag{1}$$

At the payoff time the investor incurs either gain or loss, depending on whether realization of $G(S_T) - \frac{W_0}{B(0,T)}$ is positive or negative. Problem (1) is an "isoperimetric" problem in the theory of the calculus of variations. If marginal utility, denoted U_G, is strictly decreasing, then, as shown in the appendix, the solution has the general form $G(s) = U_G^{-1}\left[\lambda \frac{\widehat{f_{s_T}(s)}}{f_{s_T}(s)}\right]$, where λ satisfies $\int U_G^{-1}\left[\lambda \frac{\widehat{f_{s_T}(s)}}{f_{s_T}(s)}\right] \widehat{f_{s_T}}(s)ds = \frac{W_0}{B(0,T)}$. Here, $f_{s_T}(s)$ and $\widehat{f}_{s_T}(s)$ are the probability density functions (p.d.f.) of S_T under the objective and martingale measures, and U_G^{-1} is the inverse function.

Let us discuss the meaning of the expression for $G(S_T)$. The quantity $\frac{\widehat{f_{s_T}(s)}}{f_{s_T}(s)}$ is called the Radon-Nikodym derivative, the stochastic discount factor, or the pricing kernel. In a model with a representative agent having additively separable utility of consumption, this term is proportional to the marginal rate of substitution in consumption: the rate at which an investor is willing to substitute consumption at time t for consumption at time t+1, i.e. $\frac{u'(c_{t+1})}{u'(c_t)}$. The term λ is a shadow price of a strategy with payoff $G(S_T)$. It is a rate of change of the expected utility at time 0 with respect to the derivative's price. The payoff is optimal if it equates marginal rate of substitution in the value of the optimal option with the marginal rate of substitution in consumption.

Example 1 *With utility function $U(G) = \frac{G^{1-\gamma}-1}{1-\gamma}$, the payoff of the optimal derivative asset is*

$$G(s) = \frac{W_0}{B(0,T)} \left\{ \frac{1}{\int \left[\frac{f(s)}{\widehat{f}(s)}\right]^{\frac{1}{\gamma}} \widehat{f_{s_T}}(s)ds} \left[\frac{f(s)}{\widehat{f}(s)}\right]^{\frac{1}{\gamma}} \right\}.$$

If the coefficient of relative risk aversion $\gamma \to 1$, i.e. an investor is characterized by a log utility, then the payoff of the optimal derivative in state s is

$$G(s) = \frac{W_0}{B(0,T)} \frac{f_{S_T}(s)}{\widehat{f_{S_T}}(s)}.$$

In this case, the payoff function is proportional to the inverse of Radon-Nikodym derivative. If the *underlying process $\{S_t\}$ is geometric Brownian motion (g.B.m.) with $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, then $\frac{S_T}{S_0} = \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T\right]$ and*

$$G(S_T) = \frac{W_0}{B(0,T)} \left(\frac{S_T}{S_0}\right)^{\frac{\mu-r}{\sigma^2}} \exp\left(-\frac{1}{2}\left[\mu + r - \sigma^2\right]\left(\frac{\mu - r}{\sigma^2}\right)T\right). \tag{2}$$

Note, that this is a continuous function of the time to expiration and the stock's price. Its value at arbitrary $t \subset [0,T]$, as shown in the appendix, can be replicated by a self-financing portfolio with $G(S_t) = p_t S_t + M_t$. Here M_t is a position in a money fund (which can be negative). One can show that the number of shares of stock in the replicating portfolio (or the optimal dynamic trading strategy) is $p_t = \frac{\partial G(S_t)}{\partial S_t} = \frac{\mu-r}{\sigma^2}\frac{1}{S_t}G(S_t)$. Expected return of this strategy is $E_0\frac{G(S_T)}{W_0} = \exp\left[\left(r + \left(\frac{\mu-r}{\sigma}\right)^2\right)T\right]$. The expected values (as of time 0) of the positions in the stock and money fund at time T are given by $E_0\left[p_T S_T\right] = \frac{(\mu-r)}{\sigma^2}\exp\left[\frac{(\mu-r)^2}{\sigma^2}T\right]$ and $E_0\left[q_T M_T\right] = -\left[\frac{\mu-\sigma^2-r}{\sigma^2}\right]\exp\left(\left(\frac{\mu-r}{\sigma}\right)^2 T\right)$. Since $E_0\left[q_T M_T\right] < 0$ when $\mu > \sigma^2 + r$ this strategy typically requires leverage. A modification of this strategy for a credit-constrained investor will be considered in the following sections.

3 Path-Dependent Optimal Derivative

In this section, I allow the terminal payoff at T to depend not only on S_T - the stock's realization at the expiration, but also on S_t, $t < T$ - the price of the stock at some intermediate time. I show that such a derivative can be viewed as a complex path-dependent structure which maximizes expected utility at T, with the terminal payoff obtained by reinvesting some intermediate derivatives at $t\,(< T)$. In addition, I establish conditions under which two payoffs – path-dependent and path-independent – are equal. If those restrictions hold, then the dynamic trading strategy that replicates or approximates the current value of a path-independent optimal derivative is an optimal strategy. Otherwise it might be suboptimal, because a trading based on a path-dependent optimal derivative might lead to a larger expected utility.

3.1 Path-Dependent Payoff

Allowing the terminal payoff $G(S_T, S_t)$ to depend on both S_T and S_t for some $t < T$ amounts to solving the following problem

$$\begin{cases} \max\limits_{G(S_T,S_t)} \int\int U\left[G\left(S_T, S_t\right)\right] f\left(S_T, S_t|S_0\right) dS_T dS_t \\ \text{s.t.} B(T,0) \int\int G\left(S_T, S_t\right) \widehat{f}\left(S_T, S_t|S_0\right) dS_T dS_t = W_0 \end{cases}$$

The Lagrangian is

$$L = U\left[G\left(S_T, S_t\right)\right] f\left(S_T, S_t|S_0\right) - \lambda \left(B(T,0) G\left(S_T, S_t\right) \widehat{f}\left(S_T, S_t|S_0\right) - W_0\right).$$

Setting $\frac{\partial L}{\partial G} = 0$ and assuming decreasing marginal utility U_G, we obtain

$$\begin{cases} G\left(S_T, S_t\right) = U_G^{-1}\left(\lambda \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right) \\ \int\int U_G^{-1}\left[\lambda \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right] \widehat{f}\left(S_T, S_t|S_0\right) dS_T dS_t = \frac{W_0}{B(T,0)} \end{cases} \tag{3}$$

Here U_G^{-1} is the inverse marginal utility function. The second equation in (3) comes from the budget constraint and yields a solution for coefficient λ. In the case of log-utility function, for example, the payoff becomes

$$G\left(S_T, S_t\right) = \frac{W_0}{B(T,0)} \frac{f\left(S_T, S_t|S_0\right)}{\widehat{f}\left(S_T, S_t|S_0\right)}. \tag{4}$$

If process $\{S_t\}$ is Markovian, then the expression $\frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}$ in (3) and (4) becomes $\frac{\widehat{f}(S_T|S_t)}{f(S_T|S_t)} \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)}$.

3.2 A Derivative with Intermediate Reinvestment

This subsection shows that a path-dependent optimal derivative can be constructed as a combination of several path-independent derivatives, such that at least one of them is optimal. To see it, consider an optimal derivative[3] $G(T,S_T,S_t)$, the payoff of which explicitly depends on time to expiration T, terminal stock's price S_T and a price S_t at an intermediate moment $t\,(<T)$. Consider also another derivative, which payoff $G\left(t,S_t\right)$, not necessarily optimal[4], realizes at $t<T$. A trader invests initially in the derivative $G\left(t,S_t\right)$ and when the payoff is obtained reinvests the proceedings into $G(T,S_T,S_t)$. This requires that the time-t value of $G(T,S_T,S_t)$ equals $G\left(t,S_t\right)$

$$G\left(t,S_t\right)=B\left(T,t\right)\int G(T,S_T,S_t)\widehat{f}\left(S_T|S_t\right)dS_T \tag{5}$$

In addition, the initial value of $G\left(t,S_t\right)$ must equal the initial investment W_0. In summary, an expected utility maximizer solves the following problem[5]

$$\begin{cases} \max\limits_{G_T,G_t}\int\int U\left[G(T,S_T,S_t)\right]f\left(S_T|S_t\right)f\left(S_t|S_0\right)dS_TdS_t \\ \text{s.t. } G\left(t,S_t\right)=B\left(T,t\right)\int G(T,S_T,S_t)\widehat{f}\left(S_T|S_t\right)dS_T \\ W_0=B\left(t,0\right)\int G\left(t,S_t\right)\widehat{f}\left(S_t|S_0\right)dS_0 \end{cases}$$

The solution is found by computing $\frac{\partial L}{\partial G_t}=0$ and $\frac{\partial L}{\partial G_T}=0$ where the Lagrangian is

$$L=U\left[G(T,S_T,S_t)\right]f\left(S_T|S_t\right)f\left(S_t|S_0\right)+\lambda_T\left(G\left(t,S_t\right)-B\left(T,t\right)G(T,S_T,S_t)\widehat{f}\left(S_T|S_t\right)\right)+$$
$$+\lambda_t\left(W_0-B\left(t,0\right)G\left(t,S_t\right)\widehat{f}\left(S_t|S_0\right)\right)$$

It yields

$$G\left(T,S_T,S_t\right)=U_G^{-1}\left[\lambda_t\frac{\widehat{f}(S_T|S_t)\widehat{f}(S_t|S_0)}{f(S_T|S_t)f(S_t|S_0)}\right]=U_G^{-1}\left[\lambda_t\frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right] \tag{6}$$

---

[3] The letter T in $G(T,S_T,S_t)$ denote that the payoff is cashed at T.

[4] Since it is not necessarily optimal, it needs not to maximize the trader's expected utility, $E_0U\left[G\left(t,S_t\right)\right]$.

[5] I could add explicitly that $G\left(t,S_t\right)$ depends on $G(T,S_T,S_t)$ because of the budget constraint $G\left(t,S_t\right)=B\left(T,t\right)\widehat{E}_tG(T,S_T,S_t)$. However, this complicates notation, and does not change the result. Additionally, those variables of $G(T,S_T,S_t)$ that affect $G\left(t,S_t\right)$ are already incorporated into the function. Also, in order to simplify the notation, I do not explicitly write that $G\left(t,S_t\right)$ depends on $(T-t)$.

where λ_t can be found from

$$W_0 = B(T,0) \int \int U_G^{-1} \left[\lambda_t \frac{\widehat{f}(S_T, S_t|S_0)}{f(S_T, S_t|S_0)} \right] \widehat{f}(S_T, S_t|S_0)\, dS_T dS_t. \tag{7}$$

Given that the form $G(T, S_T, S_t)$ is known (6), the intermediate derivative $G(t, S_t)$ is determined from the budget constraint (5).

Payoffs in (6) and (3) are the same when the process $\{S_t\}$ is Markovian. Under this very general condition a complex structure of a sequence of path-independent derivatives, as defined above, and a path-dependent optimal derivative are equivalent from investor's point of view. This finding is valuable for a profit maximizing trader in the framework where optimal path-independent derivatives are traded but the path-dependent ones are not.

3.2.1 Optimality of an Intermediate Derivative $G(t, S_t)$

If an intermediate derivative is optimal, then, as shown earlier, its payoff is given by[6] $G^o(t, S_t) = U^{-1}_{G^o(t,S_t)} \left[\lambda^o_t \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)} \right]$ with the Lagrangian multipliers λ^o_t defined by the budget constraint

$$\int U^{-1}_{G^o(t,S_t)} \left[\lambda^o_t \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)} \right] \widehat{f}(S_t|S_0)\, dS_t = \frac{W_0}{B(t,0)}.$$

It is derived by a trader who at time 0 looks forward to t.

In the previous subsection we set an intermediate derivative $G(t, S_t)$, given by the budget constraint (5). After plugging in $G(T, S_T, S_t)$ we obtain

$$G(t, S_t) = B(T,t) \int U^{-1}_{G(T,S_T,S_t)} \left[\lambda_t \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)} \right] \widehat{f}(S_T|S_t)\, dS_T$$

with λ_t taken from (7). Those two facts imply that an intermediate derivative $G(t, S_t)$ is optimal when the payoffs $G(t, S_t)$ and $G^o(t, S_t)$ are equal for every S_t. This statement is posed in the

[6] In the notation $G^o(t, S_t)$ the letter o stands for *optimal*

system

$$\begin{cases} U^{-1}_{G^o(t,S_t)}\left[\lambda^o_t \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)}\right] = B(T,t)\int U^{-1}_{G^o(T,S_T,S_t)}\left[\lambda_t \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right]\widehat{f}(S_T|S_t)\,dS_T \\ B(T,0)\int\int U^{-1}_G\left[\lambda_t \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right]\widehat{f}(S_T,S_t|S_0)\,dS_T dS_t = W_0 \\ B(t,0)\int U^{-1}_{G^o(t,S_t)}\left[\lambda^o_t \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)}\right]\widehat{f}(S_t|S_0)\,dS_t = W_0 \end{cases} \tag{8}$$

The last two equations in (8) define λ_t and λ^o_t.

Example 2 *Assuming log-utility and Markovian* $\{S_t\}$ *this system reduces to the identity* $\frac{f(S_t|S_0)}{\widehat{f}(S_t|S_0)} = \frac{f(S_t|S_0)}{\widehat{f}(S_t|S_0)}$ *with* $\lambda^o_t = \lambda_t = \frac{B(T,0)}{W_0}$. *Hence for a log-utility investor*[7] *an intermediate derivative is optimal.*

3.3 Is a Path-Dependent Derivative Always Better?

The problem of a path-dependent derivative subsumes that of the path-independent one. Hence the expected utility derived from the former is not less than that determined from the latter. In this subsection I establsih conditions under which both payoffs are equal – i.e. $G(S_T) = G(S_T, S_t)$– for every S_T, S_t. When these restrictions hold, a trading strategy based on replication of a path-independent optimal derivative is the best possible. To proceed, recall that the payoff of a path independent derivative is given by

$$\begin{cases} G(S_T) = U^{-1}_{G(S_T)}\left(\lambda_1 \frac{\widehat{f}(S_T|S_0)}{f(S_T|S_0)}\right) \\ \int U^{-1}_G\left[\lambda_1 \frac{\widehat{f}(S_T|S_0)}{f(S_T|S_0)}\right]\widehat{f}(S_T|S_0)\,dS_T = \frac{W_0}{B(T,0)} \end{cases} \tag{9}$$

and that of path-dependent one is found from

$$\begin{cases} G(S_T,S_t) = U^{-1}_{G(S_T,S_t)}\left(\lambda_2 \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right) \\ \int\int U^{-1}_G\left[\lambda_2 \frac{\widehat{f}(S_T,S_t|S_0)}{f(S_T,S_t|S_0)}\right]\widehat{f}(S_T,S_t|S_0)\,dS_T dS_t = \frac{W_0}{B(T,0)}. \end{cases} \tag{10}$$

The conditions under which both payoffs coincide are given in the following theorem.

[7] *This holds for a CRRA investor as well.*

Theorem 1 *Suppose that the optimal payoffs* $G(S_T)$, $G(S_T, S_t)$ *are unique, the process* $\{S_t\}$ *is Markovian, and utility is concave. Then* $G(S_T) = G(S_T, S_t)$ *for each* S_T, S_t *if*

$$\frac{\widehat{f}(S_T|S_t)}{f(S_T|S_t)} \frac{\widehat{f}(S_t|S_0)}{f(S_t|S_0)} = \frac{\widehat{f}(S_T|S_0)}{f(S_T|S_0)}. \tag{11}$$

Proof. Replace $\widehat{f}(S_T|S_0) = \int \widehat{f}(S_T|S_t)\, \widehat{f}(S_t|S_0)\, dS_t$ in the budget constraint (9), then plug the l.h.s. of (11) into both equations of (9) to obtain (10). Since the budget constraint equations are the same and the payoffs are unique it follows that $\lambda_1 = \lambda_2$. Similarly, we can obtain (9) from (10) and the r.h.s. of (11). ■

The conditions of the theorem hold when the underlying process is g.B.m. However, the restriction (11) is not always satisfied. For example it does not work for Heston's s.v. model. This indicates that dynamic trading strategies that replicate (or approximate) values of path-independent optimal derivatives might be suboptimal in Heston's framework.

4 Investors with Borrowing Constraints

Optimal dynamic trading typically requires leverage. However, banks may be reluctant to extend unlimited credit because of concerns over insolvency, banking regulations, or scarcity of resources. Insolvency might occur under two circumstances: (i) market incompleteness, which prevents a trader from exact replicating the current value of the optimal derivative; and (ii) a trader's utility is such that he is willing to risk insolvency. For example, if the utility were $\log(G(S_t) + C)$ where the constant $C > 0$, then the optimal payoff $G(S_T) = \frac{W_0+BC}{B}\frac{f(S_T)}{\widehat{f}(S_T)} - C$ could be negative.

To address these issues I design a dynamic trading strategy that is optimal subject to a borrowing constraint. Within the framework of complete markets, I first find the payoff $G(S_T)$ of a trader who is borrowing-constrained at the terminal moment, T. Then, the result is extended to a situation where the borrowing constraint occurs at an arbitrary intermediate time.

Such timing of credit constraints is relevant when banks report their positions discontinuously, e.g., by the end of a day, a month, etc. Obviously, the constraints do not apply in the time between required disclosures. Moreover, such a problem with simple analytical solution might be a good approximation of one where the constraints are binding continuously.

To pose the problem, we note that if replication is possible then the value of the derivative at T can be stated by $G(S_T) = \frac{\partial G(S_T)}{\partial S_T} S_T + M_T$, where $\frac{\partial G(S_T)}{\partial S_T}$ represents the number of shares in the replicating portfolio at T. If M_T, the value of the money fund investment, is negative, then the credit constraint requires $M_T \geq -L^m$ a.s. for some $L^m > 0$. As before, the initial value of the position equals the initial wealth level; i.e., $B(0,T) \int G(S_T) \widehat{f}_0(S_T) dS_T = W_0$.

The optimal constraint payoff $G(S_T)$ is the solution to

$$\begin{cases} \max\limits_G \int U(G(S_T)) f_0(S_T) dS_T \\ B(0,T) \int G(S_T) \widehat{f}_0(S_T) dS_T = W_0 \\ G(S_T) - \frac{\partial G(S_T)}{\partial S_T} S_T \geq -L^m \text{ a.s} \end{cases} \tag{12}$$

Concentrating on the case $U(\cdot) = \ln(\cdot)$, I attack the problem[8] by applying the calculus of variations. The solution has the following form, where the subscripts "b" and "n" refer to binding and non-binding parts

$$G(S_T) = \begin{cases} G^n(S_T) = \frac{1}{\lambda} \frac{f_0(S_T)}{\widehat{f}_0(S_T)} & \text{if } S_T: \ G^n(S_T) - \frac{\partial G^n(S_T)}{\partial S_T} S_T + L^m > 0 \\ G^b(S_T) = cS_T - L & \text{if } S_T: \ G^n(S_T) - \frac{\partial G^n(S_T)}{\partial S_T} S_T + L^m \leq 0 \end{cases} \tag{13}$$

Imposing continuity and smoothness of the optimal payoff $G(S_T)$ and applying the budget constraint yields solutions for the constants λ and c.

When $\{S_t\}$ is g.B.m., the unconstrained terminal payoff of an optimal derivative is given by (2).

[8]See the proof of (13) in the Appendix

However, following (13) the constrained solution is[9]

$$G\left(S_T\right)=\left[\begin{array}{l}\left\{\begin{array}{ll}G^n\left(S_T\right)=\frac{1}{[\alpha-1]}\left(\frac{S_T}{\overline{S}(L^m)}\right)^{\alpha}L^m & \text{if } S_T<\overline{S}\\ G^b(S_T)=\left(\frac{\alpha}{[\alpha-1]}\frac{S_T}{\overline{S}(L^m)}-1\right)L^m & \text{if } S_T\geq\overline{S}\\ \alpha=\frac{\mu-r}{\sigma^2}\neq 1 & \end{array}\right.\\ \left\{\begin{array}{l}G\left(S_T\right)=\left(W_0+L\right)\frac{S_T}{S_0}-L^m\\ \alpha=\frac{\mu-r}{\sigma^2}=1\end{array}\right.\end{array}\right. \tag{14}$$

The parameter $\overline{S}$ splits the domain for S_T in two parts. When $S_T < \overline{S}$ the trader's borrowing needs do not violate credit limits. In this case the optimal payoff is given by $G^n(S_T)$. On the contrary, when $S_T \geq \overline{S}$ the trader borrows as much as the constraints allow. As explained in the appendix the parameter $\overline{S}$ is found from the budget limitation. As $L^m \to 0$ (borrowing is not allowed) the value of $\overline{S} \to 0$ as well.

To illustrate this solution I present the locus of the optimal payoffs in the following figures. Figure 2a depicts the case when the credit constraint is $\frac{L^m}{W_0} = 0.6$, and Figure 2b shows the payoff when the constraint is much tighter: $\frac{L^m}{W_0} = 0.2$. The other parameters have the following values: drift, $\mu = 0.09$; dispersion, $\sigma = 0.18$; risk-free rate, $r = 0.01$; terminal time $T = 1$, and the initial stock price, $S_0 = 1$. We can see from the figures how the constraints change the shape of the terminal payoff. The heavy bold line (located between the other two) in the figure represents the optimal payoff of the *constrained* trader. Payoff of an unconstrained trader is given by a light curved line and that of the totally constrained trader (i.e. $L^m = 0$) is given by the straight (dot-dashed) line, starting from zero. The constrained payoff is between these two extremes. The unconstrained payoff is bigger when the stock does well, but in adverse circumstances or when the stock's return is just

[9]See the proof of (14) in the Appendix

slightly positive the constraint improves the outcome.

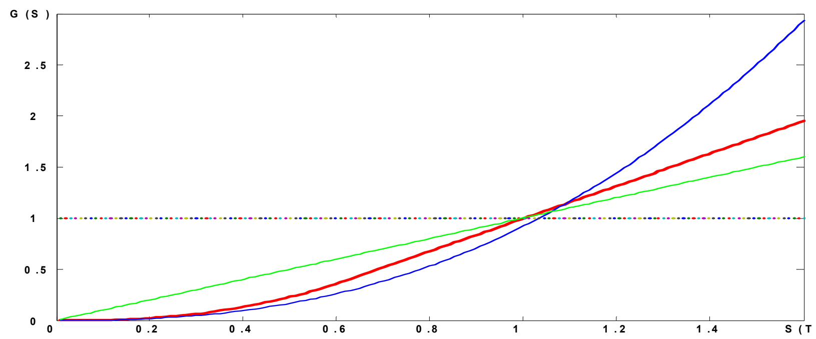

Figure $2a$ ($\frac{L}{W}=0.6$) , $\overline{S}=0.63$

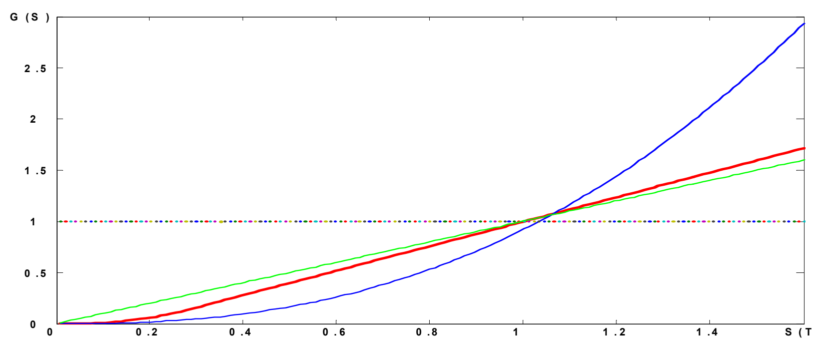

Figure $2b$ ($\frac{L}{W}=0.2$), $\overline{S}=0.28$

4.1 Intermediate Value of the Optimal Derivative.

So far we have seen how the terminal payoff, $G(S_T)$, is affected by a credit constraint at time T. This payoff can be replicated by a portfolio whose *current* value at t is $G(S_t)=B(t,T)\widehat{E_t}G(S_T)$ at $t\subset[0,T]$. In this subsection I analyze the function $G(S_t)$, and in the next discuss issues related to its replication.

For a log-utility investor the value of the replicating portfolio is

$$G(S_t)=$$
$$B(t,T)\,L^m\left\{\frac{1}{[\alpha-1]}\int\limits_0^{\overline{S}(L^m)}\left(\frac{S_T}{\overline{S}(L^m)}\right)^{\alpha}\widehat{f}(S_T|S_t)\,dS_T+\int\limits_{\overline{S}(L^m)}^{\infty}\left(\frac{\alpha}{[\alpha-1]}\frac{S_T}{\overline{S}(L^m)}-1\right)\widehat{f}(S_T|S_t)\,dS_T\right\}.$$

When $\{S_t\}$ is g.B.m. this reduces to

$$\left\{ \begin{array}{l} G\left(S_t\right) = B\left(t,T\right) L^m \left\{ \begin{array}{l} \frac{1}{[\alpha-1]} \exp\left[\left\{\left(r-\frac{1}{2}\sigma^2\right)\alpha + \frac{1}{2}\alpha^2\sigma^2\right\}\Delta T\right] \left(\frac{S_t}{\overline{S}}\right)^{\alpha} \Phi\left(N_1\right) + \\ + \frac{S_t}{\overline{S}} \frac{\alpha}{[\alpha-1]} \exp\left[r\Delta T\right]\left(1-\Phi\left(N_2\right)\right) - \left(1-\Phi\left(N_3\right)\right) \end{array} \right\} \\ N_1 = \frac{\ln\frac{\overline{S}}{S_t} - \left(\left(\alpha-\frac{1}{2}\right)\sigma^2 + r\right)\Delta T}{\sigma\sqrt{\Delta T}}, \quad N_2 = \frac{\ln\frac{\overline{S}}{S_t} - r\Delta T - \frac{1}{2}\sigma^2\Delta T}{\sigma\sqrt{\Delta T}}, \quad N_3 = \frac{\ln\frac{\overline{S}}{S_t} - r\Delta T + \frac{1}{2}\sigma^2\Delta T}{\sigma\sqrt{\Delta T}} \end{array} \right. \tag{15}$$

with $\Delta T \equiv T - t$.

The loci of the current values $G\left(S_t\right)$ of the optimal payoffs are sketched in the following figures. Figure 3a illustrates $G\left(S_0\right)$ for the short-horizon case with $T = 1$ (in which case $\overline{S} = 0.63$) and Figure 3b depicts $G\left(S_0\right)$ for the long-term case with $T = 10$ (with $\overline{S} = 0.66$).[10] The other parameters are: the constraint ratio, $\frac{L^m}{W_0} = 0.6$; drift, $\mu = 0.09$; volatility, $\sigma = 0.18$; risk free rate, $r = 0.01$; stock price, $S_0 = 1$. Each figure presents two payoff functions, the upper one shows the current at $t = 0$ value of $G\left(S_T\right)$, i.e.$G\left(S_0\right)$, discussed above, while the lower represents the terminal at $t = T$ optimal payoff, i.e. $G\left(S_T\right)$.

We see from the Figure 3a that when the time to expiration is not large there is little difference between the initial and terminal payoff functions. It might imply that violations of the credit constraint at $t < T$ are also insignificant. This possibility is investigated in the following subsection.

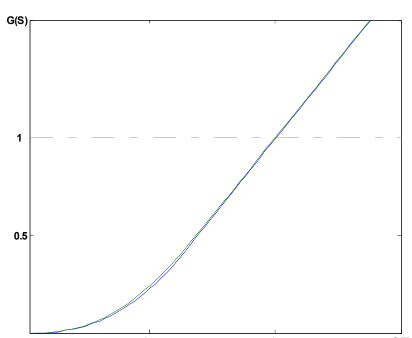

Figure 3a ($\Delta T = 1$)

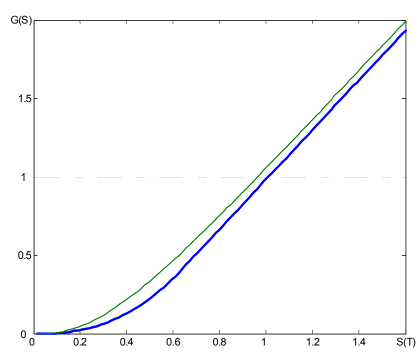

Figure 3b ($\Delta T = 10$)

[10]Observe, that $G(S_{t=0} = 1|T = 10, \frac{L}{W} = 0.6) = 1$, and $G(S_{t=T} = 1|T = 10, \frac{L}{W} = 0.6) = 0.926$. If the stock price does not change at the expiration, i.e. $S_T = S_0$, then the long position in the optimal derivative loses value. This happens because the trader borrows to invest in the stock. If the stock does not grow then the loss occurs. Moreover, the larger the time to expiration is the bigger the loss. Indeed, for this particular example we have $G(S_{t=T} = 1|T = 1) = 0.994 > G(S_{t=T} = 1|T = 10) = 0.926$

4.2 Composition of the Replicating Portfolio

Thus far, we have considered credit constraints imposed only at the terminal moment T. To see to what extent the borrowing exceeds the limit prior to T, we must find the composition of the replicating portfolio, $G(S_t) \equiv p_t S_t + M_t$. If the stock's price follows g.B.m., then the number of shares, p_t, is

$$\begin{aligned}\frac{\partial G(S_t)}{\partial S_t} &= B(t,T) L^m \frac{1}{[\alpha-1]} \exp\left[\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\alpha + \tfrac{1}{2}\alpha^2\sigma^2\right\}\Delta T\right] A_1(S_t) + \\ &+ B(t,T) L^m \left\{A_2(S_t) - A_3(S_t)\right\}\end{aligned} \tag{16}$$

where

$$A_1(S_t) = \frac{1}{S_t}\left(\frac{S_t}{\overline{S}}\right)^{\alpha}\left[\alpha\Phi(N_1) - \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\Delta T}}\exp\left(-\tfrac{1}{2}(N_1)^2\right)\right]$$

$$A_2(S_t) = \exp[r\Delta T]\frac{\alpha}{[\alpha-1]}\frac{1}{\overline{S}}\left[1 - \Phi(N_2) + \frac{1}{\sqrt{2\pi\Delta T}}\frac{1}{\sigma}\exp\left(-\tfrac{1}{2}(N_2)^2\right)\right]$$

$$A_3(S_t) = \frac{1}{\sqrt{2\pi}}\frac{1}{S_t\sigma\sqrt{\Delta T}}\exp\left(-\tfrac{1}{2}(N_3)^2\right)$$

and $\overline{S} = \overline{S}(L^m)$ is found from the budget constraint

Having these results we can determine the money fund position, $M_t = G(S_t) - S_t\frac{\partial G(S_t)}{\partial S_t}$. This is depicted in the following plots for $\frac{L^m}{W} = 0.6$, and $T = 10$. Figure 4a is for $\Delta T = 1$, and Figure 4b is for $\Delta T = 10$. In both cases the initial expiration time of the optimal derivative is $T = 10$ years.

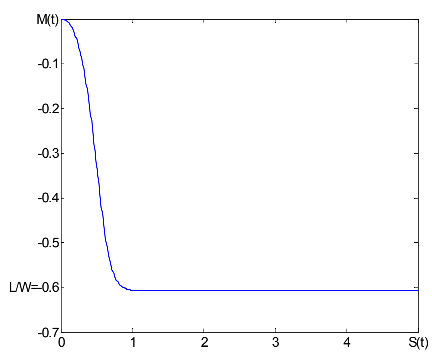

Figure $4a$ ($\Delta T = 1$)

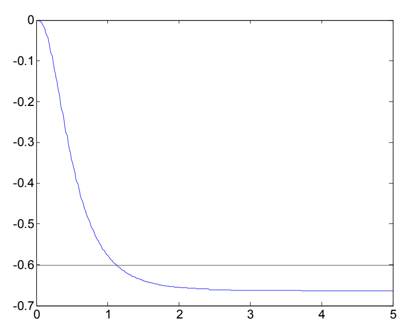

Figure $4b$ ($\Delta T = 10$)

As Figure $4a$ shows, when the remaining time to expiration is small (e.g. 10% of the derivative's life) the potential borrowing level exceeds the terminal credit constraint only slightly (1%). Figure

$4b$ shows that even when the time to expiration is large $\Delta T = 10$ the potential borrowing level barely exceeds the credit limits. These findings suggest that imposing *only terminal* credit constraints may limit the risk exposure throughout the life of the derivative.

To summarize, we have found analytically an optimal payoff for an investor who is credit-constrained at the terminal moment and demonstrated how its current value can be replicated by a self-financing dynamic strategy. The current value $G(S_t)$ is given by (15), the number of stocks p_t is indicated by (16) and the money fund position is $M_t = G(S_t) - p_t S_t$.

4.3 Borrowing Constrained at an Arbitrary Time

In this subsection I find an optimal payoff for a trader who is credit constrained at an arbitrary intermediate time $\bar{t} < T$. Recall, that a payoff of an optimal derivative is a function of time to expiration T, invested wealth W_0, and the level of the stock at the terminal moment S_T. In addition, if the borrowing constraint is exercised at $\bar{t}$, it depends on $S_{\bar{t}}$. Hence, $G_0(T, S_T, S_{\bar{t}}, W_0)$ denote the terminal payoff[11] with the initial arbitrage-free value

$$W_0 = B(0, T)\,\widehat{E}_0\left[G_0(T, S_T, S_{\bar{t}}, W_0)\right]$$

The fundamental theorem of asset pricing indicates that the current (at τ) value of $G_0(\cdot)$ is

$$g(\tau, S_\tau) = B(\tau, T)\widehat{E}_\tau\left[G_0(T, S_T, S_{\bar{t}}, W_0)\right] \tag{17}$$

The problem is defined in such a way that the amount of borrowing involved in replication of this function cannot exceed L^m at the predetermined moment $\bar{t}\,(< T)$, i.e. $g(\bar{t}, S_{\bar{t}}) - \frac{\partial g(\bar{t}, S_{\bar{t}})}{\partial S_{\bar{t}}} S_{\bar{t}} \geq -L^m$

[11] This function was denoted by $G(S_T)$ in the previous sections. The fact that $G_0(T, S_T, S_t, W_0)$ is a function of S_0 is marked by 0 subscript.

a.s. To summarize, we find the function $G_0(T, S_T, S_{\overline{t}}, W_0)$ that solves

$$\begin{cases} \max\limits_{G_0(\cdot)} E_0\left[U(G_0(T, S_T, S_{\overline{t}}, W_0))\right] \\ \text{s.t.} B\left(0,T\right) \widehat{E}_0\left[G_0(T, S_T, S_{\overline{t}}, W_0)\right] = W_0 \\ g(\tau, S_\tau) = B(\tau, T)\widehat{E}_\tau\left[G_0(T, S_T, S_t, W_0)\right] \\ g(\overline{t}, S_{\overline{t}}) - \frac{\partial g(\overline{t}, S_{\overline{t}})}{\partial S_{\overline{t}}} S_{\overline{t}} \geq -L^m \text{ a.s. for } \overline{t}\,(< T) \end{cases} \tag{18}$$

Note that if $\overline{t}$ is the moment when the credit constraint is exercised, then (subject to some regularity conditions) (17) implies

$$\begin{cases} \frac{\partial g(\overline{t}, S_{\overline{t}})}{\partial S_{\overline{t}}} = B(\overline{t}, T) \int G_0(T, S_T, S_{\overline{t}}, W_0) \frac{\partial \widehat{f}(S_T|S_{\overline{t}})}{\partial S_t} dS_T + \\ +B(t, T) \int \widehat{f}\left(S_T|S_{\overline{t}}\right) \left[\frac{\partial}{\partial S_{\overline{\overline{t}}}} G_0(T, S_T, S_{\overline{t}}, W_0)\right] dS_T \end{cases}$$

Hence, the fourth condition[12] of (18) can be written as

$$\begin{cases} C\left(S_{\overline{t}}, L\right) \equiv \\ \equiv \int \left[\left\{\widehat{f}\left(S_T|S_{\overline{t}}\right) - S_{\overline{t}} \frac{\partial \widehat{f}(S_T|S_{\overline{t}})}{\partial S_{\overline{t}}}\right\} G(\cdot) - S_{\overline{t}} \widehat{f}\left(S_T|S_{\overline{t}}\right) \frac{\partial G(\cdot)}{\partial S_{\overline{t}}} + \frac{L^m}{B(\overline{t},T)} \widehat{f}\left(S_T|S_{\overline{t}}\right)\right] dS_T \geq 0 \end{cases} \tag{19}$$

We proceed by assuming an investor with utility $U(\cdot) = \log(\cdot)$ and constructing the Lagrangian with two multipliers λ and $\xi\left(S_{\overline{t}}\right)$, .

If $S_{\overline{t}}$ is such that the constraint is binding, which means $C\left(S_{\overline{t}}, L\right) = 0$, then $\xi\left(S_{\overline{t}}\right) > 0$ and, as shown in the appendix,[13]

$$G_0(T, S_T, S_{\overline{t}}, W_0) = \frac{f\left(S_T|S_{\overline{t}}\right)}{\widehat{f}\left(S_T|S_{\overline{t}}\right)} q(S_{\overline{t}}, S_0, \xi) \tag{20}$$

The result is a product of a terminal payoff of an unconstrained optimal derivative initiated at $\overline{t}$ and expiring at T (i.e. $\frac{f(S_T|S_{\overline{t}})}{\widehat{f}(S_T|S_{\overline{t}})}$) and an optimal terminal payoff of a constraint at $\overline{t}$ optimal derivative initiated at 0 and expiring at $\overline{t}$, i.e. $q(S_{\overline{t}}, S_0, \xi)$. By plugging this solution into the definition for

[12] The credit constraint condition

[13] In the appendix it is explained that $q(S_t, S_0, \xi) = \frac{f(S_t|S_0)}{\lambda \widehat{f}(S_t|S_0) + 2\xi(S_t) + S_t \frac{d\xi(S_t)}{dS_t}}$

$g(t, S_{\bar{t}})$ we find $q(S_{\bar{t}}, S_0, \xi) = \frac{g(\bar{t}, S_t)}{B(\bar{t}, T)}$.[14] To identify an expression for $g(\bar{t}, S_{\bar{t}})$, we, as shown in the appendix, plug (20) into an equation $C(S_{\bar{t}}, L) = 0$ and obtain

$$g(\bar{t}, S_{\bar{t}}) - S_{\bar{t}} \frac{\partial g(\bar{t}, S_{\bar{t}})}{\partial S_{\bar{t}}} = -L^m \tag{21}$$

The solution to this differential equation is

$$g(\bar{t}, S_{\bar{t}}) = cS_{\bar{t}} - L^m$$

If $S_{\bar{t}}$ is such that the constraint is not binding then $\xi(S_{\bar{t}}) = 0$ and

$$G_0(T, S_T, S_{\bar{t}}, W_0) = \frac{1}{\lambda} \frac{f(S_T | S_{\bar{t}})}{\widehat{f}(S_T | S_{\bar{t}})} \frac{f(S_{\bar{t}} | S_0)}{\widehat{f}(S_{\bar{t}} | S_0)} \tag{22}$$

To summarize, $G_0(T, S_T, S_{\bar{t}}, W_0)$, the solution to the initial problem, takes the form

$$\begin{cases} G^b \equiv \frac{1}{B(\bar{t}, T)} \frac{f(S_T | S_{\bar{t}})}{\widehat{f}(S_T | S_{\bar{t}})} (cS_{\bar{t}} - L^m) & \text{if } g(\bar{t}, S_{\bar{t}}) - S_{\bar{t}} \frac{\partial g(\bar{t}, S_{\bar{t}})}{\partial S_{\bar{t}}} = -L^m \\ G^n \equiv \frac{1}{\lambda} \frac{f(S_T | S_{\bar{t}})}{\widehat{f}(S_T | S_{\bar{t}})} \frac{f(S_{\bar{t}} | S_0)}{\widehat{f}(S_{\bar{t}} | S_0)} & \text{if } g(\bar{t}, S_{\bar{t}}) - S_{\bar{t}} \frac{\partial g(\bar{t}, S_{\bar{t}})}{\partial S_{\bar{t}} t} > -L^m \end{cases}$$

The constants λ and c are found by making additional assumptions discussed in the following example.

Example 3 *Taking the g.B.m. for the underlying stock, we have shown that the optimal unconstrained payoff is given by (2). To simplify the notation we introduce two parameters,* $\alpha = \frac{\mu - r}{\sigma^2}$ *and* $\beta = -\frac{1}{2}[\mu + r - \sigma^2]\left(\frac{\mu - r}{\sigma^2}\right)$. *It can be shown that the* $\bar{t}$ *value of the payoff (given by* $g(\bar{t}, S_{\bar{t}}) = B(\bar{t}, T) \widehat{E}_{\bar{t}} G(S_T, S_{\bar{t}}, W_0)$*) is*

$$g(\bar{t}, S_{\bar{t}}) = \begin{cases} g^b(S_{\bar{t}}) = cS_{\bar{t}} - L^m & S_{\bar{t}} > \overline{S} \\ g^n(S_{\bar{t}}) = \frac{1}{\lambda} \frac{f(S_{\bar{t}} | S_0)}{\widehat{f}(S_{\bar{t}} | S_0)} & S_{\bar{t}} < \overline{S} \end{cases} \tag{23}$$

Presuming continuity and smoothness of $g(t, S_t)$*, we can figure out the constants* $c = \frac{\alpha}{\alpha - 1} \frac{L^m}{\overline{S}}$ *and*

[14] The last equation is true only at time $\bar{t}$, i.e. when the constraint is exercised.

$\frac{1}{\lambda} = \frac{L^m}{[\alpha-1]} \left(\frac{S_0}{\overline{S}}\right)^\alpha \exp\left(\beta \overline{t}\right)$. *Thus, the terminal payoff will be given by*

$$G_0(T, S_T, S_{\overline{t}}, W_0) = \begin{cases} G^b = \frac{1}{B(\overline{t},T)} \left(\frac{S_T}{S_{\overline{t}}}\right)^\alpha \exp\left[-\beta \left(T - \overline{t}\right)\right] \left(\frac{\alpha}{[\alpha-1]} \frac{S_{\overline{t}}}{\overline{S}(L^m)} - 1\right) L^m & \text{if } S_{\overline{t}} \geq \overline{S}\left(L^m\right) \\ G^n = \left(\frac{S_T}{\overline{S}(L^m)}\right)^\alpha \exp \beta \left(T - \overline{t}\right) \frac{L^m}{[\alpha-1]} & \text{if } S_{\overline{t}} < \overline{S}\left(L^m\right) \end{cases} \tag{24}$$

The constant $\overline{S}$ *can be determined numerically from*

$$\int_0^{\overline{S}} g^n\left(S_{\overline{t}}\right) \widehat{f}\left(S_{\overline{t}}|S_0\right) dS_{\overline{t}} + \int_{\overline{S}}^{\infty} g^b(S_{\overline{t}}) \widehat{f}\left(S_{\overline{t}}|S_0\right) dS_{\overline{t}} = W_0 \tag{25}$$

This completes demonstration of a methodology for setting a payoff of an optimal derivative for a hedger who is credit-constrained at some intermediate moment. Its replication by a linear stock-bond portfolio can be established by applying the standard procedure discussed earlier. This gives an optimal dynamic trading strategy for a credit constraint trader.

5 An Incomplete Markets Setting

In this section I consider an economy in which not all state variables that govern the derivative's value are traded. This precludes continuous replication of the derivative's value by a linear portfolio. Hence, an investor would like to find such an approximation to the optimal derivative that is actually attainable. To address this problem I adopt Heston's stochastic volatility model (1993). Here, the incompleteness of the market is rendered by stochastic volatility process. If the volatility were observed, then the payoff of an optimal path-independent derivative could be of several forms. Those might include $G(S_T, \sigma_T, S_0, \sigma_0)$, the function that depends on the stock price $\{S_T, S_0\}$ and the volatility $\{\sigma_T, \sigma_0\}$; as well as $G(S_T, S_0, \sigma_0)$, the payoff depends on the stock price $\{S_T, S_0\}$ and the initial volatility σ_0.[15] The latter case may be represented by a payoff of a call option in the Heston's framework.

[15] To be precise, the latter is an optimal derivative when the former is not available. Moreover, in very general terms the payoff might be $G\left(\{S_t, \sigma_t\}_{t=0}^T\right)$. We defer the discussion of this to the future research

To find an optimal payoff $G(S_T, \sigma_T, S_0, \sigma_0)$ a log-utility investor solves

$$\begin{cases} \max\limits_{G(S_T,\sigma_T,S_0,\sigma_0)} \int\int \log\left[G(S_T, \sigma_T, S_0, \sigma_0)\right] f\left(S_T, \sigma_T|\sigma_0, S_0\right) d\sigma_T dS_T \\ B\int\int G(S_T, \sigma_T, S_0, \sigma_0)\widehat{f}\left(S_T, \sigma_T|\sigma_0, S_0\right) d\sigma_T dS_T = W_0 \end{cases} \tag{26}$$

Equating the math derivative of Lagrangian to zero ($\frac{\partial L}{\partial G} = 0$) yields the solution $G(S_T, \sigma_T, S_0, \sigma_0) = \frac{W_0}{B}\frac{f(S_T,\sigma_T|\sigma_0.S_0)}{\widehat{f}(S_T,\sigma_T|\sigma_0.S_0)}$. If volatility trading were possible and could complete the market then this payoff were attainable by a linear portfolio of stock, money fund shares and an asset depending on the volatility. In an economy where the volatility is not tradable and not observable an attainable optimal derivative could feature the payoff $G(S_T, S_0)$. It is found by solving a problem similar to (26) with new payoff function[16]

$$\begin{cases} \max\limits_{G} \int\int\int \log\left[G(S_T, S_0)\right] f\left(S_T, \sigma_T|\sigma_0, S_0\right) f\left(\sigma_0\right) d\sigma_T d\sigma_0 dS_T \\ B\int\int\int G(S_T, S_0)\widehat{f}\left(S_T, \sigma_T|\sigma_0, S_0\right) f\left(\sigma_0\right) d\sigma_T d\sigma_0 dS_T = W_0 \end{cases} \tag{27}$$

The integrals can be split as

$$\begin{cases} \max\limits_{G} \int \log\left[G(S_T, S_0)\right] \left[\int f\left(S_T, \sigma_T|\sigma_0, S_0\right) d\sigma_T d\sigma_0\right] dS_T \\ B\int G(S_T, S_0)\left[\int \widehat{f}\left(S_T, \sigma_T|\sigma_0, S_0\right) d\sigma_T d\sigma_0\right] dS_T = W_0 \end{cases}$$

with the solution

$$G(S_T, S_0) = \frac{W_0}{B}\frac{\int\int f\left(S_T, \sigma_T|\sigma_0, S_0\right) f\left(\sigma_0\right) d\sigma_T d\sigma_0}{\int\int \widehat{f}\left(S_T, \sigma_T|\sigma_0, S_0\right) f\left(\sigma_0\right) d\sigma_T d\sigma_0} \tag{28}$$

To sum up, those state variables that do not contribute to the payoff function are integrated out from the joint probability density function.

In order to gain more intuition let us consider an example where the underlying follows the Heston's (1993) stochastic volatility model and an optimal derivative is represented by $G(S_T, S_0, \sigma_0)$ with unobservable initial volatility σ_0. We approximate this value by another derivative with the

[16] By assumption the initial volatility is unobserved. It is integrated out with the help of an unconditional p.d.f. $f\left(\sigma_0\right)$.

payoff $G(S_T, S_0) = \frac{W_0}{B}\frac{\int f(S_T|\sigma_0,S_0)f(\sigma_0)d\sigma_0}{\int \widehat{f}(S_T|\sigma_0,S_0)f(\sigma_0)d\sigma_0}$. According to the model, the stock dynamics is given by the following stochastic differential equations[17]

$$\begin{cases} dS = \mu^* S dt + \sigma_t S dW_1^* \\ d\sigma_t^2 = (\alpha^* - \beta^* \sigma_t^2) + \gamma\sigma_t \left(\rho dW_1^* + \sqrt{1-\rho^2} dW_2^*\right) \end{cases}$$

The volatility parameters, for both objective and risk-neutral measures, are borrowed from Chernov (2001) and presented in the table

μ	α	β	α^*	β^*	γ	ρ	r	$(T-t)$
0.05	0.014	0.93	0.0065	0.69	0.061	-0.018	0.01	5

The locus of the optimal payoff is shown in Figure 5. In general, its form depends upon the time to expiration, i.e. $(T-t)$.

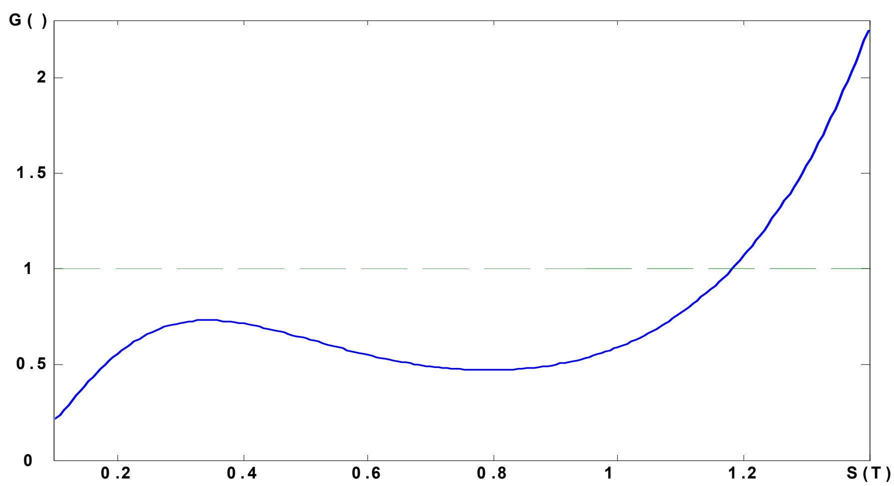

Figure 5

We now see, that unlike the optimal terminal payoff under the g.B.m., which is an increasing function of the stock, the graph of this function is characterized by a hill-like pattern in the area of

[17]The version of the model with asterisks is stated under the risk-neutral measure ($\mu^* = r$). The model without asterisks and $\mu \neq r$ is stated under the objective measure. The Brownian motions W_1 and W_2 are assumed independent.

negative return. Intuitively, investors trade off a large profit when the stock significantly gains for the opportunity to cushion loss when the stock falls.

The payoff $G(S_T, S_0)$ depends on traded state variables. Furthermore, the martingale representation theorem is applicable here. Therefore, the payoff can be replicated by a linear portfolio comprising stock and money fund shares. However, this trading strategy is suboptimal, because the conditions of the theorem 1 are not satisfied. One can show that $\frac{\hat{f}(S_T|S_t)}{f(S_T|S_t)}\frac{\hat{f}(S_t|S_0)}{f(S_t|S_0)} = \frac{\hat{f}(S_T|S_0)}{f(S_T|S_0)}$ does not hold for a process derived from the Heston model. This result means that an optimal dynamic trading strategy has to replicate a path-dependent optimal derivative, given for example by $G(\{S_t\}_{t=0}^{t=T})$. Investigation of this question is deferred to future research.

6 Conclusion

By advancing the martingale approach, this paper suggests a new view on the composition of an optimal dynamic portfolio of stock and money fund shares. This is accomplished by defining and investigating an optimal derivative along with corresponding replicating strategies. An optimal derivative features a payoff that maximizes expected utility subject to the budget constraint. Replication or approximation of its current value gives rise to an optimal dynamic trading strategy. Continuous trading allows to extend the linear scope of a static portfolio. Extension of this technique to a multi-stock setting is deferred to future research.

In this paper I introduced the concept of an optimal path-dependent derivative. In general, this might broaden the scope of payoff functions and increase the expected utility. In order to link the path-dependent and path-independent optimal derivatives I find conditions under which their optimal payoffs are the same. When those holds, trading strategies based on path-dependent derivatives do not bring additional value to the trader.

Replication of the optimal derivative's current value often requires borrowing. In practice, however, investors are limited in leverage transactions. This research addresses this issue by modifying the payoff of an optimal derivative to make it appropriate for an investor who is credit-constrained at a certain time. I also analyzed a modification of an optimal dynamic trading strategy for such an investor. Short-sale constraints will be investigated later.

In addition, this paper suggests how to approximate the terminal payoff of an optimal derivative in an incomplete markets setting by a payoff that can be replicated by traded assets. The approximation is studied in detail in the framework of the Heston stochastic volatility model where the stock is governed by a diffusion process. An extension of this question to jump processes can be accomplished in future studies.

The standard paradigm of optimal trading assumes that investor's utility is known. In applications it is often believed to be CRRA. The theoretical side of this research again starts with a utility of an investor. Together with the distribution of the underlying, this defines an optimal terminal payoff, replication of which leads to optimal dynamic trading. However, practical considerations might start directly with the terminal payoff (optimal from the investor's point of view), replication of which allows identifying trading strategies. To be more precise, we might let an investor select the optimal payoff function by providing him with information about the distribution of the underlying. Such a choice will set up a lottery, which fits trader's preferences best. Approximation of the current value of the preselected payoff function by means of a linear stock-bond portfolio gives a trading strategy that is optimal for this trader. In addition, knowing the payoff function and parameters of the distribution under the objective and risk-neutral measures allows inferring the utility function of the trader. A detailed investigation of this topic is in my plans for future research.

Appendix

Solution of (1) The problem can be rewritten as

$$\begin{cases} \max\limits_{G} \int U(G(s)) f_{S_T}(s)\, ds \\ \text{s.t. } B(0,T) \int G(s) \widehat{f}_{S_T}(s)\, ds = W_0 \end{cases}$$

Applying the methods of the calculus of variation (see F.Wan[1995]) we set the Lagrangian $L = U(G(s)) f_{S_T}(s) - \lambda \left(B(0,T) G(s) \widehat{f}_{S_T}(s) - W_0 \right)$ and plug it into the Lagrangian-Euler equation $\frac{\partial L}{\partial G} - \frac{\partial}{\partial S}\frac{\partial L}{\partial G'} = 0$ (here $G' = \frac{\partial G}{\partial S}$). Finding the constant λ from the budget constraint yields the result $G(s) = U_G^{-1}\left[B(0,T)\lambda \frac{\widehat{f_{s_T}(s)}}{f_{s_T}(s)} \right]$ with λ such that $\int U_G^{-1}\left[B(0,T)\lambda \frac{\widehat{f_{s_T}(s)}}{f_{s_T}(s)} \right] \widehat{f_{s_T}}(s) ds = \frac{W_0}{B(0,T)}$.

Replication issues Consider a T-expiring derivative originated at time 0 with terminal payoff $G(S_T)$. Its value at the current time t where $(0 \leq t \leq T)$ is given by $G(S_t) = B(t,T)\, \widehat{E_t} G(S_T)$ and depends upon two state variable: the underlying S_t and time t. Hence, the derivative $G(S_t)$ can be replicated by a self-financing portfolio comprising the stock S_t and money fund M_t as $G(S_t) = p_t S_t + q_t M_t$ at each (S_t, t). Replication assumes that

$$dG(S_t) = p_t dS_t + q_t dM_t = p_t dS_t + (G(S_t) - p_t S_t) \frac{dM_t}{M_t}$$

Plugging into it the expression for stochastic differential equation $dS_t = \mu(S_t,t)\, dt + \sigma(S_t,t)\, dW_t$ (which governs the stock S_t) and that of the money fund shares $\frac{dM_t}{M_t} = rdt$, we end up with

$$dG(S_t) = [G(S_t) r + p_t (\mu(S_t,t) - r) S_t]\, dt + p_t \sigma(S_t,t)\, dW_t \tag{29}$$

Assuming that the derivative value $G(S_t)$ is smooth enough to apply Ito's formula, i.e. continuously differentiable once with t and twice with S_t, we obtain ($G \equiv G(S_t)$)

$$dG = \left[\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial (S_t)^2} \sigma^2(S_t,t) + \frac{\partial G}{\partial S_t} \mu(S_t,t) \right] dt + \frac{\partial G}{\partial S_t} \sigma(S_t,t)\, dW_t \tag{30}$$

Canceling random parts in (29) and (30), we find that the number of the stock shares is

$$p_t = \frac{\partial G\left(S_t\right)}{\partial S_t} \tag{31}$$

This expression can be written as $p_t = B\left(t,T\right)\int G(S_T)\frac{\partial \widehat{f}(S_T|S_t)}{\partial S_t}dS_T$.

Proof of (13) The Lagrangian and the slackness conditions are

$$\begin{cases} L = \log\left(G\left(S_T\right)\right) f_0\left(S_T\right) - \lambda B_0 G\left(S_T\right)\widehat{f_0}\left(S_T\right) - \xi(S_T)\left(G(S_T) - \frac{\partial G(S_T)}{\partial S_T}S_T + L^m\right) \\ \xi(S_T)\left[G(S_T) - \frac{\partial G(S_T)}{\partial S_T}S_T + L^m\right] = 0 \text{ with } \xi(S_T) \geq 0 \end{cases}$$

This and the inequality of the problem imply that

$$\begin{cases} \xi(S_T) > 0 & \Leftrightarrow G(S_T) - \frac{\partial G(S_T)}{\partial S_T}S_T + L^m = 0 \\ \xi(S_T) = 0 & \Leftrightarrow G(S_T) - \frac{\partial G(S_T)}{\partial S_T}S_T + L^m > 0 \end{cases}$$

The condition, following from $\xi(S_T) = 0$, entails that the constraint is not binding and the task can be solved by omitting this restriction. Hence, the problem converges to

$$\begin{cases} \max\limits_{G} \int \log\left(G\left(S_T\right)\right) f_0\left(S_T\right) dS_T \\ B\left(T,0\right)\int G\left(S_T\right)\widehat{f_0}\left(S_T\right) dS_T = W_0 \end{cases}$$

with the solution $G^n\left(S_T\right) = \frac{1}{\lambda}\frac{f_0(S_T)}{\widehat{f_0}(S_T)}$. The G^n stands for the non-constrained part of the solution. Assuming the restriction holds, we find that the Lagrangian-Euler's equation is

$$G^b(S_T|S_0) = \frac{f_0\left(S_T|S_0\right)}{\frac{\partial}{\partial S_T}\left[\xi(S_T)S_T\right] + \xi(S_T) + \lambda B\left(T,0\right)\widehat{f_0}\left(S_T|S_0\right)} \tag{32}$$

This equation will be very useful later on when we approach the formula (36). However, at this stage, we find the constrained solution from the equation: $G(S_T) - \frac{\partial G(S_T)}{\partial S_T}S_T + L^m = 0$, which is given by $G^b(S_T) = cS_T - L^m$ with c to be a constant and G^b standing for the binding solution. This proves (13).

Proof of (14) Since the underlying stock follows the g.B.m. it implies that $\frac{f_0(S_T)}{\widehat{f}_0(S_T)} = \left(\frac{S_T}{S_0}\right)^{\alpha} \exp(\beta T)$ with $\alpha = \frac{\mu - r}{\sigma^2}$ and $\beta = -\frac{1}{2}\left[\mu + r - \sigma^2\right]\left(\frac{\mu - r}{\sigma^2}\right) T$. In this case there is such $\overline{S}$ that

$$G(S_T) = \begin{cases} \frac{1}{\lambda B(t,0)} \left(\frac{S_T}{S_0}\right)^{\alpha} \exp \beta T & \text{if } S_T < \overline{S} \\ c S_T - L^m & \text{if } S_T \geq \overline{S} \end{cases} \tag{33}$$

The constant c is identified by assuming continuity of the payoff function everywhere, including the point $\overline{S}$. This means that the unconstrained and constrained solutions are equalized at $\overline{S}$ (i.e. no jumps in the payoff function are allowed). Thus, $G^n\left(\overline{S}\right) = G^b(\overline{S})$. Moreover, we assume that the payoff function is smooth, i.e. its math derivative exists at any point. It implies that at $\overline{S}$ we have $\frac{\partial}{\partial S_t} G^n\left(\overline{S}\right) = \frac{\partial}{\partial S_t} G^b(\overline{S})$. These preconditions give us

$$\begin{cases} \lambda = \frac{1}{L^m B(T,0)} \left(\frac{\overline{S}}{S_0}\right)^{\alpha} [\alpha - 1] \exp \beta T \\ c = \frac{\alpha}{[\alpha - 1]} \frac{L^m}{\overline{S}} \end{cases}$$

such that we obtain (14). Inserting $G(S_T)$ from (33) into the budget constraint

$$B(T,0) \int G(S_T) f(S_T | S_0) = W_0$$

we find an equation which establishes $\overline{S}$

$$\frac{1}{[\alpha - 1]} \int_0^{\overline{S}} \left(\frac{S_T}{\overline{S}}\right)^{\alpha} \widehat{f}(S_T)\, dS_t + \int_{\overline{S}}^{\infty} \left(\frac{\alpha}{[\alpha - 1]} \frac{S_T}{\overline{S}} - 1\right) \widehat{f}(S_T)\, dS_t = \frac{W_0}{B(T,0) L} \tag{34}$$

We can show that the l.h.s. of (34) is a decreasing function of $\overline{S}$. In a very general case, though, there might be several solutions. Then, we pick the one that maximizes an expected utility. We find $\overline{S}$ by observing that (34) can be written as

$$\begin{cases} \frac{1}{[\alpha-1]}\left(\frac{S_0}{\overline{S}}\right)^{\alpha}\exp\left[\left(r-\frac{1}{2}\sigma^2\right)\alpha t+\frac{1}{2}\alpha^2\sigma^2 t\right]\cdot\Phi\left(\frac{\ln\frac{\overline{S}}{S_0}-\left(\left(\alpha-\frac{1}{2}\right)\sigma^2+r\right)t}{\sigma\sqrt{t}}\right)+ \\ \frac{S_0}{\overline{S}}\frac{\alpha}{[\alpha-1]}\exp\left[rt\right]\cdot\left\{1-\Phi\left(\frac{\ln\frac{\overline{S}}{S_0}-rt-\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right)\right\}-\left\{1-\Phi\left(\frac{\ln\frac{\overline{S}}{S_0}-rt+\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right)\right\} \\ =\frac{W_0}{B(T,0)L} \end{cases} \tag{35}$$

Since the l.h.s. of (35) is a decreasing function of $\overline{S}$ with the range $[\infty, 0]$ it can be readily solved for $\overline{S}$.

Proof of (20) The Lagrangian of the problem (18) takes the form

$$\begin{cases} L=\log(G(T,S_T,S_t))f\left(S_T|S_t\right)f\left(S_t|S_0\right)- \\ -\lambda\left(G(T,S_T,S_t)\widehat{f}\left(S_T|S_t\right)\widehat{f}\left(S_t|S_0\right)-\frac{W_0}{B(T,0)}\right)-\xi\left(S_t\right)C\left(S_t,L\right) \end{cases}$$

The slackness condition and the requirement that $\xi\left(S_t\right)\geq 0$ imply that

$$\begin{cases} C\left(S_t,L\right)>0 & \text{if } \xi\left(S_t\right)=0 \\ C\left(S_t,L\right)=0 & \text{if } \xi\left(S_t\right)>0 \end{cases}$$

The solution to the Lagrangian-Euler's equation $\frac{\partial L}{\partial G}=\frac{d}{dS_t}\frac{\partial L}{\partial G'}$ (here $G'\equiv\frac{\partial G}{\partial S_t}$) is

$$G\left(S_T,S_t,T\right)=\frac{f\left(S_T|S_t\right)f\left(S_t|S_0\right)}{\lambda\left(\widehat{f}\left(S_T|S_t\right)\widehat{f}\left(S_t|S_0\right)\right)+\xi\left(S_t\right)\left\{\widehat{f}\left(S_T|S_t\right)-S_t\frac{\partial\widehat{f}(S_T|S_t)}{\partial S_t}\right\}+\frac{d}{dS_t}\left[S_t\widehat{f}\left(S_T|S_t\right)\xi\left(S_t\right)\right]}$$

If the constraint is binding (i.e. $\xi\left(S_t\right)>0$ and $C\left(S_t,L\right)=0$), the expression above can be reduced by taking derivatives and simplifying

$$G=\frac{f\left(S_T|S_t\right)}{\widehat{f}\left(S_T|S_t\right)}\cdot\frac{f\left(S_t|S_0\right)}{\lambda\widehat{f}\left(S_t|S_0\right)+2\xi\left(S_t\right)+S_t\frac{d\xi(S_t)}{dS_t}}\equiv\frac{f\left(S_T|S_t\right)}{\widehat{f}\left(S_T|S_t\right)}g(S_t,S_0,\xi) \tag{36}$$

which explains the formula (20). Observe that the function $g(S_t,S_0,\xi)$ resembles that in (32).

Proof of (21) We plug an expression for G (i.e.20) into $C(S_t, L)$ and by simplifying obtain

$$\begin{cases} g(S_t,S_0,\xi) - S_t g(S_t,S_0,\xi)\int \frac{f(S_T|S_t)}{\widehat{f}(S_T|S_t)}\frac{\partial}{\partial S_t}\widehat{f}\,(S_T|S_t)\,dS_T - \\ S_t\frac{\partial g(S_t,S_0,\xi)}{\partial S_t} + S_t g(S_t,S_0,\xi)\left[\int \frac{f(S_T|S_t)}{\widehat{f}(S_T|S_t)}\frac{\partial}{\partial S_t}\widehat{f}\,(S_T|S_t)\,dS_T\right] + \frac{L}{B(t,T)} = 0 \end{cases}$$

Along the way we presumed regularity conditions required for

$$\frac{\partial}{\partial S_t}\int f\,(S_T|S_t)\,dS_T = \int \frac{\partial}{\partial S_t} f\,(S_T|S_t)\,dS_T$$

Since this expression is zero, the result follows.

Heston model application The c.d.f. and p.d.f. of the underlying stock in the Heston model takes the form ($\tau = T - t$):

$$\begin{cases} \Pr[S_T < y] = \frac{1}{2} - \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{i\xi}(y)^{-i\xi}\Psi_{\log S_T}(i\xi;\cdot)\,d\xi \\ f_{S_T}(y|S_t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{y}(y)^{-i\xi}\Psi_{\log S_T}(i\xi;\log S_t\cdot)\,d\xi \\ \Psi\left(i\xi;\log S_t,\sigma_t^2,\tau\right) = \exp\left[r\tau + g(\tau,i\xi) + h(\tau,i\xi)\,\sigma_t^2 + i\xi\log S_t\right] \end{cases}$$

where $\Psi_{\log S_T}(i\xi;\log S_t\cdot)$ is a characteristic function of $\log S_T$. Hence, the p.d.f.'s under the objective and risk-neutral measures can be expressed as (we take the drift $\mu = const$)

$$\begin{cases} f\,(S_T|S_t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{1}{S_T}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[g(\tau,i\xi) + h(\tau,\xi)\,\sigma_t^2\right]d\xi \\ \widehat{f}\,(S_\tau|S_t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{1}{S_T}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[r\tau + \widehat{g}(\tau,i\xi) + \widehat{h}(\tau,\xi)\,\sigma_t^2\right]d\xi \text{ ok} \end{cases} \tag{37}$$

such that ratio of the p.d.f.'s is

$$\frac{f\left(S_T|S_t,\sigma_t^2\right)}{\widehat{f}\left(S_T|S_t,\sigma_t^2\right)} = \frac{\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[g(\tau,i\xi) + h(\tau,i\xi)\,\sigma_t^2\right]d\xi}{\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[r\tau + \widehat{g}(\tau,i\xi) + \widehat{h}(\tau,i\xi)\,\sigma_t^2\right]d\xi}$$

The functions $g(\tau, i\xi)$and $h(\tau, i\xi)$ are

$$\begin{cases} h(\tau) = \widehat{h}(\tau) = \frac{B-D}{\gamma^2}\frac{\exp(D\tau)-1}{1-Q\exp(D\tau)} \\ \widehat{g}(\tau) = (i\xi - 1) r\tau + \frac{\alpha}{\gamma^2}\left[(D-B)\tau - 2\ln\left(\frac{1-Q\exp(D\tau)}{1-Q}\right)\right] \\ g(\tau) = i\xi\mu\tau + \frac{\alpha}{\gamma^2}\left[(D-B)\tau - 2\ln\left(\frac{1-Q\exp(D\tau)}{1-Q}\right)\right] \end{cases}$$

Under the objective measure, the drift is $\mu \neq r$ and the variables are defined as

$$\begin{cases} z = i\xi & A = -\frac{z}{2} + \frac{z^2}{2} \quad B = z\rho\gamma - \beta \\ D = \sqrt{B^2 - 2A\gamma^2} & Q = \frac{B-D}{B+D} \end{cases}$$

Hence, after plugging those into the function for the p.d.f. we get

$$G(S_T, S_0) = \frac{W_0}{B}\frac{\int\limits_0^{\infty}\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[g(\tau, i\xi) + h(\tau, \xi)v\right] f_{\sigma_t^2}(v)\,d\xi dv}{\int\limits_0^{\infty}\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\exp\left[r\tau + \widehat{g}(\tau, i\xi) + \widehat{h}(\tau, \xi)v\right]\widehat{f}_{\sigma_t^2}(v)\,d\xi dv} \tag{38}$$

To integrate out volatility we need to find an unconditional (or the steady state) p.d.f. for σ_t^2, i.e. $f_{\sigma_t^2}(v)$. In the Heston's model it is defined by the s.d.e.

$$d\sigma_t^2 = \left(\alpha - \beta\sigma_t^2\right)dt + \gamma\sigma_t dW_t$$

with the solution

$$\begin{cases} f_{\sigma_t^2}(v) = \frac{w^z}{\Gamma(z)}v^{z-1}\exp(-wv) \\ w = \frac{2\beta}{\gamma^2} \;\; z = \frac{2\alpha}{\gamma^2} \end{cases} \tag{39}$$

After plugging (39) into the nominator and the denominator of (38) and reducing we find

$$f(S_T|S_t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\left(\frac{w}{w - h(\tau, i\xi)}\right)^z \exp g(\tau, i\xi)\,d\xi$$

Hence, the optimal terminal payoff is

$$G\left(S_T|S_t\right)=\frac{W_0}{B}\frac{\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\left(\frac{w}{w-h(\tau,i\xi)}\right)^{z}\exp g\left(\tau,i\xi\right)d\xi}{\int\limits_{-\infty}^{+\infty}\left(\frac{S_T}{S_t}\right)^{-i\xi}\left(\frac{\widehat{w}}{\widehat{w}-\widehat{h}(\tau,i\xi)}\right)^{z}\exp \widehat{g}\left(\tau,i\xi\right)d\xi}$$

References

Bertsimas, D., Kogan, L., and Lo, A., “Pricing and Hedging Derivative Securities in Incomplete Markets: An e-Arbitrage approach”, 2001, *Operations Research* 49, 372-397.

Brennan, M., Solanki, R.,”Optimal portfolio insuarance”, 1981, *The Journal of financial and quantitative analysis*, 16(3), 279-300.

Carr, P. and Madan, D. ”Optimal positioning in derivative securities”, 2001, *Quantitative Finance*, 1, 19-37.

Chernov, M. and Ghysels, E. ”A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation”, 2000, *Journal ofFinancial Economics, 56(3), 407-458*

Cohrane, J. ”Asset Pricing”, 2001, Princeton University Press.

Cox, J. and Huang, C. ”Optimal Consumption and Portfolio Policies when Asset Prices follow a Diffusion Process”, 1989, *Journal of Economic Theory*, 49, 33-83.

Epps, W. ”A Quick Way to Calculate Put Prices from Characteristic Function of Underlying”, 2003, mimeo, University of Virginia .

Harrison, J. and Kreps, D. ”Martingales and arbitrage in multiperiod securities markets”, 1979, *Journal of Economic Theory* 20(3), 381-408.

Harrison, J. and Pliska, S. ”Martingales and stochastic integrals in the theory of continuous trading”, 1981, *Stochastic Processes and Applications* 11, 215-260.

He, H. and Pearson, N. ”Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case”, 1991, *Journal of Economic Theory*, 54, 259-304.

Heston, S. ”A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”, 1993, *The Review of Financial Studies*, 6(2) 327-343.

Karatzas, I., Lehoczky, J., Shreve S. ”Optimal portfolio and consumption decisions for a ”small investor” on a finite horizon”, 1987, *SIAM Journal of Control and Optimization*, 25, 1557-1586.

Leland, H. ”Who should buy portfolio insurance?”, 1980, *Journal of Finance* 35(2), 581-594.

Merton, R. ”Optimum consumption and portfolio rules in a continuous time model”, 1971, *Journal of Economic Theory*, 3, 373-413.

Pliska, S. "A stochastic calculus model of continuous trading: optimal portfolios",1986, *Mathematics of Operations Research*, 11, 371-82.

Tepla, L. "Optimal portfolio policies with borrowing and shortsale constraints", 2000, *Journal of Economics Dynamic and Control* 24, 1623-1639.

Zhao, Y., Haussmann,U., Ziemba W. "A Dynamic Investment Model with Control on the Portfolio's Worst Case Outcome", 2003, *Mathematical Finance*, 13(4), 481 - 501.

Wan, F. "The calculus of variation and its application", 1995.